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# Reduction number of links of irreducible varieties

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**Abstract**

The reductions of an ideal $I$ give a natural pathway to the properties of $I$, with the advantage of having fewer generators. In this paper we primarily focus on a conjecture about the reduction exponent of links of a broad class of primary ideals. The existence of an algebra structure on the Koszul and Eagon–Northcott resolutions is the main tool for detailing the known cases of the conjecture. In the last section we relate the conjecture to a formula involving the length of the first Koszul homology modules of these ideals. © 1997 Elsevier Science B.V.

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## 1. Introduction

Blowing up a variety along a subvariety is a very important kind of transformation in Algebraic Geometry, especially in the process of resolution of singularities. Algebraically, this is encoded in the following diagram:

$$\begin{array}{ccc} \mathrm{Proj}(\mathrm{gr}_I(R)) & \hookrightarrow & \mathrm{Proj}(R[It]) \\ \downarrow & & \downarrow \varphi \\ V(I) & \hookrightarrow & \mathrm{Spec}(R) \end{array} \qquad (1)$$

where $R$ is a commutative Noetherian local ring and $I$ is one of its ideals. The existence of the morphism $\varphi$ in (1), which is an isomorphism outside $\varphi^{-1}(V(I)) = \mathrm{Proj}(\mathrm{gr}_I(R))$,

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justifies the attention paid to find conditions for the *Rees algebra* and the *associated graded ring* of $I$, collectively referred to as *blowup algebras*,

$$R[It]=\bigoplus_{i=0}^{\infty} I^it^i, \quad \text{and} \quad \mathrm{gr}_I(R)=\bigoplus_{i=0}^{\infty} I^i/I^{i+1},$$

to be Cohen–Macaulay.

In what follows, it is described how methods from Linkage Theory (see [10]) are providing advances on the Cohen–Macaulayness issue. Recall that two ideals $I$ and $L$ of height $g$ in a Cohen–Macaulay local ring $R$ are said to be *directly linked* if there exists a regular sequence $z = z_1, \ldots, z_g \subseteq I \cap L$ such that $I = (z) : L$ and $L=(z):I$. Furthermore, they are said to be *geometrically linked* if they are unmixed ideals, without common components and $I \cap L = z$.

Also, a *reduction* of an ideal $I$ (see [11]) is an ideal $J \subseteq I$ such that, for some nonnegative integer $r$, the equality $I^{r+1} = JI^r$ holds. The smallest such integer is the reduction number $r_J(I)$ of $I$ relative to $J$. If the residue field of $R$ is infinite, then minimal reductions always exist and their minimal number of generators does not depend on the minimal reduction. This number is called *analytic spread of I* and is always greater than or equal to the height of the ideal $I$; in case of equality the ideal $I$ is said to be *equimultiple*.

The specific question motivating this paper is to find conditions for a link of an irreducible variety to have Cohen–Macaulay blowup algebras. Since irreducible varieties correspond to the locus of primary ideals, the central object of this study consists of ideals of the form $I = J : P$, where $J = (z)$ is an ideal generated by a regular sequence of length $g$ inside $P$ and $P$ is a primary ideal of height $g$. Of course, one cannot expect a positive answer to the previous question if $P$ is any primary ideal. The first issue is then to single out families of primary ideals for which good results are expected. The first two natural choices are: ($a$) links of prime ideals; ($b$) links of symbolic powers of prime ideals.

Two previous papers (see [5, 6]) were devoted to the study of the first family of ideals and the key point in order to have a good understanding of that family was to show that links of prime ideals are equimultiple with reduction number one. Indeed, the counterpart of a low reduction exponent is, in general, a very high depth of the blowup algebras associated with the ideals (see [5, Corollary 2.4; 6, Theorem 3.1, Corollary 3.2]). In its broadest formulation one has

**Theorem 1.1.** *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring and let $J=(z_1,\ldots,z_g)$ be an ideal generated by a regular sequence inside a prime ideal $\mathfrak{p}$ of height $g$. If we set $I=J:\mathfrak{p}$ then $I^2=JI$ if one of the following two conditions holds*:

($L_1$) *$R_\mathfrak{p}$ is not a regular local ring*;

($L_2$) *$R_\mathfrak{p}$ is a regular local ring of dimension at least 2 and two of the $z_i$'s lie in $\mathfrak{p}^{(2)}$*.

Theorem 1.1 is sharp in the sense that if condition ($L_2$) is not satisfied then $I$ is generically a complete intersection, hence it does not admit any proper reduction. A straightforward generalization of Theorem 1.1 reads as follows:

**Theorem 1.2.** *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring and let $P$ be an unmixed radical ideal of $R$ of codimension $g$, such that $R_{\mathfrak{p}}$ is not a regular local ring for any prime $\mathfrak{p}$ minimal over $P$. Then every link $I$ of $P$ is equimultiple with reduction number one.*

The focus of this paper is on the second family of ideals, called *k-iterated links of prime ideals* (see Definition 2.2 below); the main conjecture, whose formulation is inspired by the one of Theorem 1.1, is about their reduction number and it can be stated as follows:

**Conjecture 1.3.** *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring and $\mathfrak{p}$ a prime ideal of height $g \geq 2$. Let $J = (z_1, \ldots, z_g)$ be an ideal generated by a regular sequence contained in $\mathfrak{p}^{(s)}$, where $s$ is a positive integer $\geq 2$. For any $k = 1, \ldots, s$ set $I_k = J : \mathfrak{p}^{(k)}$. Then*

$$I_k^2 = J I_k$$

*if one of the following two conditions holds:*

($IL_1$) *$R_{\mathfrak{p}}$ is not a regular local ring;*

($IL_2$) *$R_{\mathfrak{p}}$ is a regular local ring and two of the $z_i$'s lie in $\mathfrak{p}^{(s+1)}$.*

Note, however, that in the main result of this paper ($IL_2$) is replaced by the weaker assumption

($IL_2^*$) *$R_{\mathfrak{p}}$ is a regular local ring and either its dimension is at least 3 or $J \subseteq \mathfrak{p}^{(s+1)}$ whenever its dimension is 2.*

We now describe the results of this manuscript. The main result of Section 2 is Theorem 2.4, which establishes the above conjecture when: (a) condition ($IL_2^*$) holds; (b) the *initial forms* (see [3, p. 179]) of the generators of $J$ form a regular sequence on the associated graded ring of $\mathfrak{p}$. The approach is ideal-theoretic as in the case of [5]. The first step is the reduction to the case of $\mathfrak{m}$-primary ideals; then, standard mapping cone arguments lead to a detailed analysis of the structure of the ideals $I_k$. The main tool for this analysis is the existence of an algebra structure on the Koszul and the Eagon–Northcott resolutions; this accounts for hypothesis (a). Hypothesis (b) is, on the other hand, necessary in order to use a well-established criterion due to Valabrega –Valla (see [16, Corollary 2.7]).

In Section 3, instead, $R_{\mathfrak{p}}$ is only supposed to be Gorenstein. The approach, here, is more homological rather than ideal-theoretic. More precisely, we relate the length of $I_k^2/J I_k$ to the length of the kernel $\delta(I_k)$ of the natural surjection from $S_2(I_k)$ onto $I_k^2$. Furthermore, we conjecture that the sum of these two lengths equals $\binom{n_k+1}{2}$, where $n_k = \lambda(\mathfrak{m}^{k-1}/\mathfrak{m}^k)$. This fact is already known in the case $k = 1$ (see [6]); we now show that it holds in the case $k = 2$ as well. The proof is based on a formula for the length of the first Koszul homology module of $I_2$ (see Theorem 3.7).

As a general rule, we will provide the basic definitions and the references for all the results that are used; the reader desiring more details should then consult the invaluable books of Bruns and Herzog [3], Matsumura [8], and the excellent monograph of Vasconcelos [17].

## 2. Iterated links of prime ideals in regular local rings

A useful method to test whether a link of an ideal is equimultiple with reduction number one is stated in the next lemma.

**Lemma 2.1.** *Let $(R,\mathfrak{m})$ be a local ring and let $I$ be one of its ideals. If $I$ is an equimultiple ideal of height $g$ with reduction number* 1, *i.e.,* $I^2 = JI$ *where $J$ is an ideal generated by a regular sequence $z_1,\dots,z_g$, then*

$$I \subseteq J : I \quad \text{and} \quad (J:I)J = (J:I)I. \tag{1}$$

*Conversely, the conditions in* (1) *are sufficient to guarantee that $I$ has reduction number* 1 *with respect to $J$ provided $I = J : (J : I)$.*

**Proof.** From $I^2 = JI \subseteq J$ it follows that $I \subseteq J : I$, and this takes care of the first condition in (1) — $J$ does not need to be generated by a regular sequence. Since $(J : I)J$ is clearly contained in $(J : I)I$, it is enough to show that any $z \in (J : I)I$ can be written as an element in $(J : I)J$. But $(J : I)I \subseteq J$, so that there exist $\alpha_1,\dots,\alpha_g$ such that

$$z = \sum_{i=1}^{g} \alpha_i z_i. \tag{2}$$

The proof will be complete provided it is shown that $\alpha_i$ is in $J : I$ for all $i$ in the range $1,\dots,g$. Pick $a \in I$ and consider the element $za$ in $(J : I)I^2 = (J : I)JI \subseteq J^2$; Hence

$$za = \sum_{i=1}^{g} \beta_i z_i, \tag{3}$$

for some $\beta_1,\dots,\beta_g$ in $J$. A comparison between Eq. (3) and $a$ times Eq. (2) yields the following identity:

$$\sum_{i=1}^{g} (\alpha_i a - \beta_i) z_i = 0,$$

which says that $(\alpha_i a - \beta_i) z_i \equiv 0 \bmod (z_1,\dots,\widehat{z_i},\dots,z_g)$, i.e., $\alpha_i a \in J$ for all $i$ as claimed.

The proof of the converse is similar and can be found, essentially, in [5, Theorem 2.2]. □

**Definition 2.2.** Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring and $\mathfrak{p}$ a prime ideal of height $g$. Let $J$ be an ideal generated by a regular sequence contained in $\mathfrak{p}^{(s)}$, where $s$ is a positive integer. Given any integer $k = 1,\ldots,s$, define $I_k = J : \mathfrak{p}^{(k)}$ and call it *kth iterated link* of $\mathfrak{p}$, as $I_k = I_{k-1} : \mathfrak{p}$.

**Lemma 2.3.** *In the situation described in Definition* 2.2 *one has that*

(a) *if* $J \subseteq \mathfrak{p}^s$ *then for* $k = 1,\ldots,s$ $I_k = J : \mathfrak{p}^k = J : \mathfrak{p}^{(k)}$;

(b) *if* $R_\mathfrak{p}$ *is a Gorenstein local ring, then for* $k = 1,\ldots,s$ $I_k$ *and* $\mathfrak{p}^{(k)}$ *are directly linked via* $J$.

**Proof.** (a) Set $H_k = J : \mathfrak{p}^k$. Then $\mathfrak{p}^k I_k \subseteq \mathfrak{p}^{(k)} I_k \subseteq J$ says that $J : \mathfrak{p}^{(k)} = I_k \subseteq H_k = J : \mathfrak{p}^k$. Hence, it suffices to show the equality at the associated primes of the ideal $I_k$, which are contained in the ones of $J$; hence they all have height $g$. The conclusion now follows as $\mathfrak{p}^{(k)} = \mathfrak{p}^k R_\mathfrak{p} \cap R$. This takes care of (a).

(b) This is a basic property of Linkage Theory (see [10]). □

Next, we formulate the main result of this paper; the rest of this section will be entirely devoted to its proof, which will be achieved in two steps. First, using a well-established method (see [5, 6]), the problem is reduced to the maximal case, i.e., to ideals of the form $I_k = J : \mathfrak{m}^k$. Subsequently, using Remark 2.7 together with a mapping cone construction which produces a free resolution of $I_k$, one is able to outline the set-theoretic structure of the ideal $I_k$. More precisely, in Proposition 2.8 it is shown that $I_k$ is made up of $J$ and extra $\binom{d+k-2}{d-1}$ generators that lie in a sufficiently large power of $\mathfrak{m}$ (here $d$ denotes the dimension of the ring $R$). The notions of order and initial form of an element are central in the following result so they are formalized next. The *order* $o(r)$ of any element $r$ in $R$ is defined as the unique positive integer $q$ such that $r \in \mathfrak{m}^q$ but $r \notin \mathfrak{m}^{q+1}$. If $o(r) = q$, then the *initial form* of $r$ is the residue class of $r$ in $\mathfrak{m}^q/\mathfrak{m}^{q+1}$.

**Theorem 2.4.** *Let* $(R,\mathfrak{m})$ *be a Cohen–Macaulay local ring and* $\mathfrak{p}$ *a prime ideal of height* $g \geq 2$ *such that* $R_\mathfrak{p}$ *is a regular local ring. Let* $J = (z_1,\ldots,z_g)$ *be an ideal generated by a regular sequence contained in* $\mathfrak{p}^{(s)}$, *where s is a positive integer* $\geq 2$, *and assume that the initial forms of the* $z_i$*'s form a regular sequence on* $\mathrm{gr}_\mathfrak{p}(R)$. *Then for any* $k = 1,\ldots,s$, *the ideal* $I_k = J : \mathfrak{p}^{(k)}$ *satisfies*

$$I_k^2 = J\, I_k$$

*if the height of* $\mathfrak{p}$ *is at least* 3 *or if* $J \subseteq \mathfrak{p}^{(s+1)}$ *whenever the height of* $\mathfrak{p}$ *is* 2.

**Remark 2.5** (*Reduction to the maximal case*). In Theorem 2.4 $\mathfrak{p}$ may be assumed to be the maximal ideal of the ring.

**Proof.** As seen in [5,6], it is enough to establish the equality $I_k^2 = J\, I_k$ at the associated primes of $J\, I_k$. From the exact sequence

$$0 \to J/J\, I_k \to R/J\, I_k \to R/J \to 0$$

and the fact that $J/J\, I_k = J/J^2 \otimes R/I_k \simeq (R/I_k)^g$, it follows that the associated primes of $J\, I_k$ are contained in those of $R/J$ and $R/I_k$. On the other hand, $I_k$ and $\mathfrak{p}^{(k)}$ are directly linked via $J$ so that $\operatorname{Ass}(R/I_k) \subseteq \operatorname{Ass}(R/J)$. This implies that the associated primes of $JI_k$ must have height $g$. Therefore for $\mathfrak{q} \in \operatorname{Ass}(R/J)$ either $\mathfrak{q} = \mathfrak{p}$ or $\mathfrak{q} \not\supseteq \mathfrak{p}$. In the second case $(I_k)_\mathfrak{q} = J_\mathfrak{q}$ and the proof is complete; the first case instead is the case of a link with a power of the maximal ideal. Clearly, all the other conditions localize. □

**Remark 2.6.** A resolution $\mathscr{F} = \{\mathscr{F}_i\}$ with differentials $\partial = \{d_i\}$ of a cyclic module $R/H$ is called an *algebra resolution* if there is a graded associative multiplication (called a DG-*algebra structure*) $\mathscr{F} \otimes_R \mathscr{F} \to \mathscr{F}$ lifting the usual product on $R/H$ and satisfying

(i) $x_i x_j = (-1)^{ij} x_j x_i$ for $x_i \in \mathscr{F}_i$ and $x_j \in \mathscr{F}_j$;

(ii) $d_{i+j}(x_i x_j) = (d_i x_i) x_j + (-1)^i x_i (d_j x_j)$ for $x_i \in \mathscr{F}_i$ and $x_j \in \mathscr{F}_j$.

If also $\partial \mathscr{F} \subset \mathfrak{m}\mathscr{F}$ then $\mathscr{F}$ is called a *minimal algebra resolution*. The best known example is the Koszul resolution, which has the structure of an exterior algebra on a free module.

**Remark 2.7.** Set $d$ to be the dimension of the ring $R$. Let $\mathscr{K}$ and $\mathscr{E}\mathscr{N}$ denote the Koszul resolution of $J = (z_1, \ldots, z_d)$ and the Eagon–Northcott resolution of $\mathfrak{m}^k$, respectively. The natural inclusion $J \subseteq \mathfrak{m}^k$ induces the following comparison diagram:

$$
\begin{array}{lccccccccccccc}
\mathscr{E}\mathscr{N}: & 0 & \longrightarrow & \mathscr{F}_{k,d} & \xrightarrow{\delta_{d-1}} & \mathscr{F}_{k,d-1} & \xrightarrow{\delta_{d-2}} & \cdots\cdots & \xrightarrow{\delta_1} & \mathscr{F}_{k,1} & \xrightarrow{\delta_0} & \mathscr{F}_{k,0} & \longrightarrow & 0 \\
 & & & \uparrow u_{k,d} & & \uparrow u_{k,d-1} & & & & \uparrow u_{k,1} & & \uparrow u_{k,0} & & \\
\mathscr{K}: & 0 & \longrightarrow & \mathscr{K}_d & \xrightarrow[d_d]{} & \mathscr{K}_{d-1} & \xrightarrow[d_{d-1}]{} & \cdots\cdots & \xrightarrow[d_2]{} & \mathscr{K}_1 & \xrightarrow[d_1]{} & \mathscr{K}_0 & \longrightarrow & 0
\end{array}
\tag{5}
$$

where

$$\mathscr{K}_t = \bigwedge^t R^d \quad \text{for } t \geq 0,$$

$$\mathscr{F}_{k,0} = \bigwedge^k R^k,$$

$$\mathscr{F}_{k,t} = D_{t-1}((R^k)^*) \otimes_R \bigwedge^{k+t-1} R^{d+k-1} \quad \text{for } t \geq 1.$$

Here, $D_{t-1}((R^k)^*)$ denotes the degree $t-1$ component of the *divided powers algebra* of the module $(R^k)^*$.

As the Koszul and the Eagon–Northcott resolutions both have an associative, commutative, differential graded structure and the Koszul resolution is a universal object, it is enough to define the map $u_k = \{u_{k,t}\}$ on a basis of the Koszul resolution. To this end, let $E_1, \ldots, E_d$ be a basis of $\mathscr{K}_1 \simeq R^d$ and $e_1, \ldots, e_{d+k-1}$ be a basis of $R^{d+k-1}$. Then

$$
\begin{aligned}
u_{k,d}(E_1 \wedge \cdots \wedge E_d) &= u_{k,1}(E_1) \bigstar \cdots \bigstar u_{k,1}(E_d) \\
&= \prod_{j=1}^{d} \sum_{\substack{h=1 \\ |A_{h,k}|=k}}^{\binom{d+k-2}{d-1}} \alpha_{hj}(1 \otimes e_{A_{h,k}}),
\end{aligned}
\tag{6}
$$

where $u_{k,1}$ is the map that rewrites the $z_i$'s in terms of the basis elements of $\mathfrak{m}^k$. Also, $A_{h,k} = (a_1, \ldots, a_k)$ is an ordered $k$-upla with each $a_i$ an element of $\{1, \ldots, d+k-1\}$ and $e_{A_{h,k}} = e_{a_1} \wedge \cdots \wedge e_{a_k}$.

**Proposition 2.8.** *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$. Let $J = (z_1, \ldots, z_d)$ be an ideal generated by a regular sequence in $\mathfrak{m}^s$, where $s$ is a positive integer $\geq 2$. For any $k = 1, \ldots, s$ set $I_k = J : \mathfrak{m}^k$ and $q_i = o(z_i)$. Then one can write $I_k = (J, J_k)$ with*

$$\mu(J_k) = \binom{d+k-2}{d-1} \quad \text{and} \quad J_k \subseteq \mathfrak{m}^{q-k+1},$$

*where $q = \sum_{j=1}^{d} q_j - d$ if the dimension of $R$ is at least 3, or $q = q_1 + q_2$ if the dimension of $R$ is 2 and $J \subseteq \mathfrak{m}^{s+1}$.*

**Proof.** From (5), the fact that $I_k = J : \mathfrak{m}^k$, and [17, Theorem 4.1.2] it follows that the beginning of a free resolution of $R/I_k$ is given by

$$\to (\mathscr{K}_{d-2} \oplus \mathscr{F}_{k,d-1})^* \xrightarrow{\partial_{d-1}^*} (\mathscr{K}_{d-1} \oplus \mathscr{F}_{k,d})^* \xrightarrow{\partial_d^*} (\mathscr{K}_d)^* \to R/I_k \to 0.$$

Since $\partial_d = (-d_d, -u_{k,d})^{\mathrm{T}}$, one has

$$I_k = \partial_d^*((\mathscr{K}_{d-1} \oplus \mathscr{F}_{k,d})^*) = (J, -u_{k,d}(\mathscr{K}_d)^{\mathrm{T}}).$$

Moreover, $\mathscr{F}_{k,d} \simeq R^{\binom{d+k-2}{d-1}}$ hence $J_k = -u_{k,d}(\mathscr{K}_d)^{\mathrm{T}}$ has $\mu_k = \binom{d+k-2}{d-1}$ generators, i.e.,

$$I_k = (J, b_{k,1}, \ldots, b_{k,\mu_k}). \tag{7}$$

Using (6), the fact that $\alpha_{hj} \in \mathfrak{m}^{q_j-k}$ (or $\alpha_{hj} \in \mathfrak{m}^{q_j+1-k}$ if $d = 2$), and $d-1$ times the explicit formula for the product $\bigstar$ given in [14, Section 6] — which is valid regardless of the characteristic of $R$ — one concludes that the $b_{k,j}$'s belong to the power $\mathfrak{m}^e$, where

$$e = \sum_{j=1}^{d} q_j - dk + (d-1)(k-1) = \sum_{j=1}^{d} q_j - d - k + 1$$

(or $e = q_1 + q_2 - k + 1$ if $d = 2$). Thus, one has that $J_k \subseteq \mathfrak{m}^{q-k+1}$, where $q = \sum_{j=1}^{d} q_j - d$ (or $q = q_1 + q_2$ if $d = 2$). □

**Remark 2.9.** Note that $q - k + 1 \geq q_d + (d-1)s - d - k + 1 = q_d + (d-1)(s-1) - k$. Thus $q - k + 1 \geq q_d + s - k$ if $d \geq 3$, and $q - k + 1 = q_2 + s - k + 1 > q_2 + s - k$ if $d = 2$.

We now have all the tools to complete the proof of Theorem 2.4.

**Proof of Theorem 2.4.** By a result of Valabrega–Valla (see [16, Corollary 2.7]), the elements of $J$ satisfy the following equality for any $n \geq 1$:

$$\mathfrak{m}^n \cap J = \sum_{i=1}^{d} \mathfrak{m}^{n-q_i} z_i \,,$$

where $q_i = o(z_i)$. As in a local ring regular sequences permute, one may assume that $q_1 \leq \cdots \leq q_d$. Hence for any $n \geq 1$ it follows that

$$\mathfrak{m}^n \cap J \subseteq \mathfrak{m}^{n-q_d} J \,. \tag{8}$$

In particular, (8) used with $n = q_d + s$ gives the inclusion

$$\mathfrak{m}^{q_d+s} \cap J \subseteq \mathfrak{m}^s J \subseteq \mathfrak{m}^k J \,. \tag{9}$$

This is enough to prove the assertion. Indeed, Proposition 2.8 and Remark 2.9 imply that $\mathfrak{m}^k J_k \subseteq \mathfrak{m}^{q_d+s}$. On the other hand, from the definition of $I_k$ it follows that $\mathfrak{m}^k J_k \subseteq \mathfrak{m}^k I_k \subseteq J$; hence (9) says that $\mathfrak{m}^k J_k \subseteq \mathfrak{m}^k J$ and so

$$\mathfrak{m}^k I_k = \mathfrak{m}^k (J, J_k) = \mathfrak{m}^k J + \mathfrak{m}^k J_k \subseteq \mathfrak{m}^k J \,.$$

This completes the proof of the theorem as both the conditions in Lemma 2.1 (see (1)) are satisfied. □

**Question 2.10.** Let $I = J : H$, where $H$ is a perfect ideal with a minimal free resolution having a DG-algebra structure (see [9] for a list of such ideals). Is it true that under some mild hypotheses $I^2 = JI$? Mild hypotheses on $J$ and $H$ are needed as the next example shows.

Let $R$ be a two dimensional regular local ring with maximal ideal $\mathfrak{m} = (x, y)$ and pick any positive integer $r \geq 2$. The $\mathfrak{m}$-primary ideals $J = (x^{r+1}, y^{r+1})$ and $H = (x, y^r)$ yield the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & R & \xrightarrow{\binom{-y^r}{x}} & R^2 & \xrightarrow{(x\,y^r)} & R & \longrightarrow & R/H & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \wedge^2\varphi} & & \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \\
0 & \longrightarrow & R & \xrightarrow[\binom{-y^{r+1}}{x^{r+1}}]{} & R^2 & \xrightarrow[(x^{r+1}\,y^{r+1})]{} & R & \longrightarrow & R/J & \longrightarrow & 0
\end{array}$$

where $\varphi = \begin{pmatrix} x^r & 0 \\ 0 & y \end{pmatrix}$ is the matrix that writes the generators of $J$ in terms of the ones of $H$, and $\wedge^2 \varphi = \det(\varphi) = x^r y$. Both the deleted resolutions have a DG-algebra structure, as they are Koszul resolutions. However, the link $I = J : H = (x^{r+1}, y^{r+1}, x^r y)$ satisfies $r_J(I) = r \geq 2$.

## 3. Iterated links of prime ideals in Gorenstein rings

In this section we approach Conjecture 1.3 from a different perspective, namely the one of [6]. In the case $k = 1$, it turns out that $I_1 = J : \mathfrak{m} = (J, v_1, \ldots, v_t)$, where $\overline{v}_1, \ldots, \overline{v}_t$ are the generators of the socle of $R/J$ and $t$ is the *type* of $R$ (see [3, Definition 1.2.15]). Moreover, one also has the formula (see [5, Remark 2.7])

$$\lambda(I_1^2/JI_1) + \lambda(\delta(I_1)) = \binom{t+1}{2}, \tag{10}$$

where $\delta(I_1)$ is the kernel of the natural surjection from $S_2(I_1)$ onto $I_1^2$; (10) is the key tool in the proof of [6, Theorem 2.1], since in the Gorenstein case $t = 1$ and the case $\delta(I_1) = 0$ can not occur provided that either condition ($\mathrm{L}_1$) or ($\mathrm{L}_2$) is satisfied.

Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension $d$ and $J$ an ideal of $R$ generated by a regular sequence $(z_1, \ldots, z_d)$ contained in $\mathfrak{m}^s$, for some positive integer $s \geq 2$. Set $I_k = J : \mathfrak{m}^k$ for $k = 1, \ldots, s$. The goal of this section is to point out the analogue of (10) for the $I_k$'s with $k \geq 2$. More precisely, we will show in Proposition 3.3 that

$$\lambda(I_k^2/JI_k) + \lambda(\delta(I_k)) = \binom{n_k+1}{2}$$

is equivalent to

$$\lambda(H_1(I_k)) = n_k\lambda(R/I_k) + \binom{n_k+1}{2} - \lambda(R/\mathfrak{m}^k),$$

where $n_k = \lambda(\mathfrak{m}^{k-1}/\mathfrak{m}^k)$. We end the section by showing that the latter formula holds in the case $k = 2$ (see Theorem 3.7). These are the technical results of the manuscript and they are interesting as they uncover an error in a formula that appears in [1] and saying, in our terminology, that

$$\lambda(H_1(I_k)) \leq n_k\lambda(R/I_k).$$

As, in what follows, $R$ need not be a regular local ring but simply Gorenstein, one cannot use Proposition 2.8 in order to describe the structure of $I_k$, since $R/\mathfrak{m}^k$ does not necessarily have a finite projective resolution. This is taken care of in Proposition 3.2 below. Some additional assumptions will be needed.

**Remark 3.1.** From now on, it will always be required that $\mathfrak{m}^k : \mathfrak{m} = \mathfrak{m}^{k-1}$ for $k = 3, \ldots, s$; this is always satisfied if, e.g., $\operatorname{depth}(\operatorname{gr}_{\mathfrak{m}}(R)) \geq 1$. Also, note that $d \geq 2$ (or, in the general case, $g \geq 2$) is needed as the next example shows.

Let $R = k[X,Y,Z]/(Y^2 - XZ, X^3 - Z^2) = k[x,y,z]$, where $x$, $y$, and $z$ denote the images of $X$, $Y$, and $Z$ modulo $(Y^2 - XZ, X^3 - Z^2)$. It can be shown (see [12]) that $R$ is a Gorenstein ring of dimension 1 with $\operatorname{gr}_{\mathfrak{m}}(R)$ Gorenstein as well. Let $J = (x^4)$ be a regular sequence in $\mathfrak{m}^4 = (x,y,z)^4$ and consider $I_3 = J : \mathfrak{m}^3$; a computation using the computer system *Macaulay* shows that $I_3^2 \neq JI_3$.

**Proposition 3.2.** *Let $(R,\mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 2$ and let $J = (z_1,\dots,z_d)$ be an ideal generated by a regular sequence contained in $\mathfrak{m}^s$, where $s$ is a positive integer $\geq 2$. For $k = 1,\dots,s$ set $I_k = J : \mathfrak{m}^k$ and, for $k = 3,\dots,s$, assume that $\mathfrak{m}^k : \mathfrak{m} = \mathfrak{m}^{k-1}$. Then*

(a) *$I_k/I_{k-1}$ is an $R/\mathfrak{m}$-vector space of dimension $n_k$, where $n_k = \lambda(\mathfrak{m}^{k-1}/\mathfrak{m}^k)$. Hence $I_k = (I_{k-1}, b_1,\dots,b_{n_k})$, with $\widehat{b}_1,\dots,\widehat{b}_{n_k}$ minimal generators of $\widehat{I}_k = I_k/I_{k-1}$;*

(b) *$I_k = (J, b_1,\dots,b_{n_k})$, with $\overline{b}_1,\dots,\overline{b}_{n_k}$ minimal generators of $\overline{I}_k = I_k/J$;*

(c) *$J$ is among the minimal generators of $I_k$ if and only if $\mu(I_k) = d + n_k$;*

(d) *$I_{k-1} = \mathfrak{m}I_k + J$.*

**Proof.** One can write $I_k = I_{k-1} : \mathfrak{m}$ so that $\mathfrak{m}I_k \subseteq I_{k-1}$. This shows that $I_k/I_{k-1}$ is an $R/\mathfrak{m}$-vector space. As $\mathrm{Hom}_{R/J}(-,R/J)$ is a length preserving functor (see [3, Theorem 3.2.12]), from the equality $\mathrm{Hom}_{R/J}(R/\mathfrak{m}^k, R/J) = J : \mathfrak{m}^k/J = I_k/J$ it follows that

$$\lambda(I_k/J) = \lambda(\mathrm{Hom}_{R/J}(R/\mathfrak{m}^k, R/J)) = \lambda(R/\mathfrak{m}^k)\,. \tag{11}$$

Hence

$$\lambda(I_k/I_{k-1}) = \lambda(I_k/J) - \lambda(I_{k-1}/J) = \lambda(\mathfrak{m}^{k-1}/\mathfrak{m}^k) = n_k\,. \tag{12}$$

This completes the proof of (a).

(b) follows from the fact that $\mu(I_k/J)$ equals the type of $R/\mathfrak{m}^k$ (see [3, Proposition 3.3.11(c,i)]). More precisely,

$$\begin{aligned}\mu(I_k/J) &= \dim(\mathrm{Hom}_R(R/\mathfrak{m}, R/\mathfrak{m}^k))\\ &= \dim(\mathfrak{m}^k : \mathfrak{m}/\mathfrak{m}^k) = \lambda(\mathfrak{m}^{k-1}/\mathfrak{m}^k) = n_k\,.\end{aligned}$$

Note that in the last equality we used the hypothesis that $\mathfrak{m}^{k-1} : \mathfrak{m} = \mathfrak{m}^{k-1}$.

Finally, (c) and (d) readily follows from (a) and (b). □

**Proposition 3.3.** *Let $(R,\mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 2$ and let $J = (z_1,\dots,z_d)$ be an ideal generated by a regular sequence contained in $\mathfrak{m}^s$. For $k = 1,\dots,s$ set $I_k = J : \mathfrak{m}^k$ and, for $k = 3,\dots,s$, assume that $\mathfrak{m}^k : \mathfrak{m} = \mathfrak{m}^{k-1}$. Furthermore, assume that the $z_i$'s are among the minimal generators of $I_k$ and let $n_k = \lambda(\mathfrak{m}^{k-1}/\mathfrak{m}^k)$. Then the following two results hold:*

(a) *the length of the symmetric square $S_2(I_k/J)$ is given by*

$$\lambda(S_2(I_k/J)) = \lambda(I_k^2/J\,I_k) + \lambda(\delta(I_k)) = \rho\,, \tag{13}$$

*where $\delta(I_k)$ is the kernel of the natural surjection from $S_2(I_k)$ onto $I_k^2$;*

(b) *the length of the first Koszul homology module $H_1(I_k)$ is given by*

$$\lambda(H_1(I_k)) = n_k\lambda(R/I_k) + \rho - \lambda(R/\mathfrak{m}^k)\,. \tag{14}$$

*Moreover, $\rho = \binom{n_k+1}{2}$ if and only if $S_2(I_k/J)$ is an $R/\mathfrak{m}$-vector space.*

**Proof.** As in the proof of [5, Remark 2.7] one has the short exact sequence

$$0 \to \delta(I_k) \to S_2(I_k/J) \to I_k^2/J\,I_k \to 0\,,$$

which leads to the equation

$$\lambda(S_2(I_k/J)) = \lambda(I_k^2/J\,I_k) + \lambda(\delta(I_k))\,. \tag{15}$$

If we set $\lambda(S_2(I_k/J)) = \rho$, (14) follows from (13), [6, Proposition 2.2], and (11).

By Proposition 3.2 one has that $I_{k-1} = \mathfrak{m}I_k + J$. Hence the last assertion follows from

$$S_2(I_k/I_{k-1}) = S_2(I_k/(\mathfrak{m}I_k + J)) = S_2(I_k/J \otimes R/\mathfrak{m}) \simeq S_2(I_k/J) \otimes R/\mathfrak{m}$$

and the fact that $I_k/I_{k-1}$ is an $R/\mathfrak{m}$-vector space of dimension $n_k$. □

**Remark 3.4.** Note that the conjectured value for $\rho = \lambda(S_2(I_k/J))$ does not depend on $I_k$ nor on $J$ but on $\mathfrak{m}$ alone. This follows from the fact that $I_k/J$ is isomorphic to the canonical module of $R/\mathfrak{m}^k$.

**Remark 3.5.** It also has to be pointed out that (14) can be derived in a more direct manner. It is well-known that $H_1(I_k) \simeq H_1(\bar{I}_k)$, hence one may first reduce modulo $J$. Moreover, the standard short exact sequence

$$0 \to \mathscr{B}_1 \to \mathscr{Z}_1 \to H_1(\bar{I}_k) \to 0 \tag{16}$$

implies that $\lambda(H_1(I_k)) = \lambda(H_1(\bar{I}_k)) = \lambda(\mathscr{Z}_1) - \lambda(\mathscr{B}_1)$. Let

$$0 \to \mathscr{Z}_1 \to (R/J)^{n_k} \to I_k/J \to 0$$

be a minimal presentation of $I_k/J$. Then

$$\lambda(\mathscr{Z}_1) = n_k\lambda(R/J) - \lambda(I_k/J) = n_k\lambda(R/I_k) + (n_k - 1)\lambda(R/\mathfrak{m}^k)\,, \tag{17}$$

as, by (11), $\lambda(I_k/J) = \lambda(R/\mathfrak{m}^k)$. On the other hand $\mathscr{B}_1$ fits in the short exact sequence

$$0 \to \mathscr{B}_1 \to (I_k/J)^{n_k} \to S_2(I_k/J) \to 0\,, \tag{18}$$

which is essentially in [13] (see also [15, proof of Theorem 3.1]), so that

$$\lambda(\mathscr{B}_1) = n_k\lambda(I_k/J) - \lambda(S_2(I_k/J)) = n_k\lambda(R/\mathfrak{m}^k) - \rho\,.$$

Finally, (16), (17), and (18) give that

$$\begin{aligned}\lambda(H_1(I_k)) &= n_k\lambda(R/I_k) + (n_k - 1)\lambda(R/\mathfrak{m}^k) - \big(n_k\lambda(R/\mathfrak{m}^k) - \rho\big)\\ &= n_k\lambda(R/I_k) + \rho - \lambda(R/\mathfrak{m}^k)\,,\end{aligned}$$

as in Proposition 3.3.

A technical fact is needed in order to prove Theorem 3.7 below; even though we only need this result in the case $k = 2$, we prove it in full generality.

**Lemma 3.6.** *Let $(R,\mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 2$ and let $J = (z_1,\dots,z_d)$ be an ideal generated by a regular sequence contained in $\mathfrak{m}^s$, where $s$ is a positive integer $\geq 2$. For $k = 1,\dots,s$ set $I_k = J : \mathfrak{m}^k$ and, for $k = 3,\dots,s$, assume that $\mathfrak{m}^k : \mathfrak{m} = \mathfrak{m}^{k-1}$. Then there exists a non degenerate bilinear form*

$$\theta_k : \mathfrak{m}^{k-1}/\mathfrak{m}^k \times I_k/I_{k-1} \to R/\mathfrak{m},$$

*which determines a well-defined, non-singular square matrix whose size is $n_k = \lambda(\mathfrak{m}^{k-1}/\mathfrak{m}^k)$.*

**Proof.** Given $\widetilde{m} = m + \mathfrak{m}^k \in \mathfrak{m}^{k-1}/\mathfrak{m}^k$ and $\widehat{l} = l + I_{k-1} \in I_k/I_{k-1}$ one has that $ml \equiv cv \pmod{J}$, as $\mathfrak{m}^{k-1}I_k \subseteq I_1$ and $I_1 = (J,v)$. Thus a bilinear form $\theta_k$ can be defined, up to an element of $\mathfrak{m}$, by

$$\theta_k(\widetilde{m},\widehat{l}) = c\,. \tag{19}$$

It is routine to verify that $\theta_k$ is a well-defined bilinear form. Let us prove, instead, the non-degeneracy of $\theta_k$. To say that $\theta_k(\widetilde{m},\widehat{l}) \equiv 0 \pmod{\mathfrak{m}}$ for all $\widehat{l} \in I_k/I_{k-1}$ means that $ml \in J$ for all $l \in I_k$. But $\mathfrak{m}^k = J : I_k$, so that $m \in \mathfrak{m}^k$. On the other hand, $\theta_k(\widetilde{m},\widehat{l}) \equiv 0 \pmod{\mathfrak{m}}$ for all $\widetilde{m} \in \mathfrak{m}^{k-1}/\mathfrak{m}^k$ implies that $ml \in J$ for all $m \in \mathfrak{m}^{k-1}$; hence $l \in I_{k-1}$, since $I_{k-1} = J : \mathfrak{m}^{k-1}$, so that $l \equiv 0 \pmod{I_{k-1}}$.

Both $\mathfrak{m}^{k-1}/\mathfrak{m}^k$ and $I_k/I_{k-1}$ are $R/\mathfrak{m}$-vector spaces of dimension $n_k$; hence, $\theta_k$ can be written in terms of fixed bases of those two vector spaces. Let $\{\widetilde{x}_1,\dots,\widetilde{x}_{n_k}\}$ and $\{\widehat{b}_1,\dots,\widehat{b}_{n_k}\}$ be bases of $\mathfrak{m}^{k-1}/\mathfrak{m}^k$ and $I_k/I_{k-1}$ respectively. With respect to those two bases let $\widetilde{m} = (a_1,\dots a_{n_k})$ and $\widehat{l} = (d_1,\dots d_{n_k})$. Finally, letting $x_ib_j \equiv c_{ij}v \pmod{J}$, one has

$$\begin{aligned} ml &\equiv \left(\sum_{i=1}^{n_k} a_ix_i\right)\left(\sum_{j=1}^{n_k} d_jb_j\right) = \sum_{i=1}^{n_k} a_i\sum_{j=1}^{n_k} d_jx_ib_j \mod J \\ &\equiv \left(\sum_{i=1}^{n_k} a_i\sum_{j=1}^{n_k} d_jc_{ij}\right) v\,. \end{aligned}$$

Thus, the $c$ that appears in (19) is $\sum_i a_i \sum_j d_jc_{ij}$; in matrix form, the former expression can be written as (note that the $c_{ij}$'s are seen modulo $\mathfrak{m}$)

$$(a_1 \cdots a_{n_k}) \begin{pmatrix} c_{11} & \cdots & c_{1n_k} \\ \vdots & & \vdots \\ c_{n_k1} & \cdots & c_{n_kn_k} \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_{n_k} \end{pmatrix}.$$

Since $\theta_k$ is a nondegenerate quadratic form, $(c_{ij})$ is a nonsingular matrix. □

**Theorem 3.7.** *Let $(R,\mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 2$ and let $J = (z_1,\dots,z_d)$ be an ideal generated by a regular sequence contained in $\mathfrak{m}^s$, where $s$ is a positive integer $\geq 2$. For $k = 1,2$ set $I_k = J : \mathfrak{m}^k$ and assume that $z_1,\dots,z_d$ are* *among the minimal generators of $I_2$. Then the length of the first Koszul homology module of $I_2$ is*

$$\lambda(H_1(I_2)) = n\lambda(R/I_2) + \binom{n+1}{2} - \lambda(R/\mathfrak{m}^2),$$

*where $n = n_2 = \lambda(\mathfrak{m}/\mathfrak{m}^2)$.*

**Proof.** According to Remark 3.5, it will be enough to show that

$$\lambda(\mathscr{B}_1) = n\lambda(R/\mathfrak{m}^2) - \binom{n+1}{2} = n(1+n) - \binom{n+1}{2} = \binom{n+1}{2}, \tag{20}$$

as $\lambda(R/\mathfrak{m}^2) = \lambda(R/\mathfrak{m}) + \lambda(\mathfrak{m}/\mathfrak{m}^2) = 1 + n$. Note that $\mathscr{B}_1$ is the submodule of $\overline{R}^n$ generated by the $n$-uples $\overline{u}_{ij}$ whose nonzero components are in the $i$th and in the $j$th positions and are given by $\overline{b}_j$ and $-\overline{b}_i$, respectively, i.e.,

$$\overline{u}_{ij} = (\ldots, \overline{b}_j, \ldots, -\overline{b}_i, \ldots);$$

hence, $\mu(\mathscr{B}_1) = \binom{n}{2}$. In order to show (20), we use the short exact sequence

$$0 \to \overline{\mathfrak{m}}\mathscr{B}_1 \to \mathscr{B}_1 \to \mathscr{B}_1/\overline{\mathfrak{m}}\mathscr{B}_1 \to 0.$$

Observe that both $\mathscr{B}_1/\overline{\mathfrak{m}}\mathscr{B}_1$ and $\overline{\mathfrak{m}}\mathscr{B}_1$ are $R/\mathfrak{m}$-vector spaces and that (20) follows provided one shows that

$$\lambda(\mathscr{B}_1/\overline{\mathfrak{m}}\mathscr{B}_1) = \binom{n}{2} \quad \text{and} \quad \lambda(\overline{\mathfrak{m}}\mathscr{B}_1) = n.$$

However $\lambda(\mathscr{B}_1/\overline{\mathfrak{m}}\mathscr{B}_1) = \mu(\mathscr{B}_1) = \binom{n}{2}$, thus it remains to show that $\lambda(\overline{\mathfrak{m}}\mathscr{B}_1) = n$. As in the proof of Lemma 3.6, one can consider $\overline{\mathfrak{m}} = (\overline{x}_1, \ldots, \overline{x}_n)$ and write $\overline{x_i b_j} = \overline{c_{ij} v}$. Thus a typical element of $\overline{\mathfrak{m}}\mathscr{B}_1$ is of the form

$$\begin{aligned}\overline{m u}_{ij} &= \left(\sum_k \overline{a_k x_k}\right) \overline{u}_{ij} = \sum_k \overline{a_k x_k u_{ij}} = \sum_k \overline{a}_k(\ldots, \overline{x_k b}_j, \ldots, -\overline{x_k b}_i, \ldots) \\ &= \sum_k \overline{a_k v}(\ldots, \overline{c}_{kj}, \ldots, -\overline{c}_{ki}, \ldots).\end{aligned}$$

So $\overline{\mathfrak{m}}\mathscr{B}_1$ is a subspace of a $n$-dimensional $R/\mathfrak{m}$-vector space and it is generated by the following $n\binom{n}{2}$ vectors

$$\overline{v}(\ldots, \overline{c}_{kj}, \ldots, -\overline{c}_{ki}, \ldots),$$

where $i, j, k = 1, \ldots n$. One can identify $\overline{\mathfrak{m}}\mathscr{B}_1$ as a subspace of $(R/\mathfrak{m})^n$ by identifying the previous vectors with the $n$-uples (defined modulo $R/\mathfrak{m}$)

$$(\ldots, c_{kj}, \ldots, -c_{ki}, \ldots).$$

Now, if the subspace spanned by those vectors was a proper subspace then it would be contained in some hyperplane, i.e., those vectors would all satisfy an equation of the form

$$\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n = 0, \tag{21}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a nontrivial $n$-upla. Fix $i, j$ and substitute the vector $(\ldots, c_{kj}, \ldots, -c_{ki}, \ldots)$ in (21) for each $X_k$. This leads to the equations

$$\alpha_1 c_{1j} + \alpha_2 c_{2j} + \cdots + \alpha_n c_{nj} = 0,$$

$$\alpha_1 c_{1i} + \alpha_2 c_{2i} + \cdots + \alpha_n c_{ni} = 0.$$

Letting $i$ and $j$ vary between 1 and $n$ yields the following homogeneous system of linear equations

$$\begin{cases} \alpha_1 c_{11} + \alpha_2 c_{21} + \cdots + \alpha_n c_{n1} = 0, \\ \vdots \\ \alpha_1 c_{1n} + \alpha_2 c_{2n} + \cdots + \alpha_n c_{nn} = 0. \end{cases}$$

But $\alpha \neq 0$ contradicts the nonsingularity of the matrix $(c_{ij})$ proved in Lemma 3.6. Thus, $\lambda(\overline{\mathfrak{m}\mathscr{B}_1}) = n$. $\square$

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